Formation of soliton trains in Bose-Einstein condensates by temporal Talbot effect

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We study the recent observation of formation of matter-wave soliton trains in Bose-Einstein condensates. We emphasize the role of the box-like confinement of the Bose-Einstein condensate and find that there exist time intervals for the opening the box that support the generation of real solitons. When the box-like potential is switched off outside the existing time windows, the number of peaks in a train changes resembling missing solitons observed in the experiment. Our findings indicate that a new way of generating soliton trains in condensates through the temporal, matter-wave Talbot effect is possible.

The notion of soliton belongs to the most popular concepts in physics. Any localized and long-living structure is readily called a soliton. Solitons are identified with solutions of nonlinear equations and appear in many contexts of science ranging from physics (nonlinear optics, hydrodynamics, particle physics, etc.) to molecular biology. Solitons are formed because of the existence of nonlinear interaction in the system which cancels the dispersion and hence allows for the propagation of shape preserving objects.

In the case of dilute atomic quantum gases the nonlinearity is determined by the effective interaction between atoms that can be both repulsive or attractive. For repulsive uniform condensates the appropriate nonlinear equation (i.e. the Gross-Pitaevskii equation) predicts dark soliton (a hole in the density associated with a phase jump) as a solution [1]. Such excitations of Bose-Einstein condensates have been already observed in experiments with trapped alkali atoms [2].

Generating bright solitons in atomic quantum gases is more difficult task because it requires working with attractive condensates. Due to a collapse, in such samples the number of atoms is limited and small (see Ref. [3]). This obstacle has been overcome in two ways. In the first one, a large repulsive condensate is formed and then the interactions are changed from repulsive to attractive by using the Feshbach resonance technique [4, 5], whereas in the second attempt [6] the optical lattice and the notion of negative effective mass are utilized.

The goal of this Letter is twofold. First, we focus on the experiment of Ref. [4] where the trains of bright solitons are generated in attractive condensate of ^{7}Li atoms. Existing theoretical explanations involve quantum phase fluctuations [7] or modulational instability [8, 9] as the main mechanism leading to the observed structures. None of these papers, however, attempt to model the experiment of Ref. [4]. In our Letter we report on calculations that correspond to the same values of all parameters as in Ref. [4] with realistic three-dimensional geometry. Secondly, we concentrate on the role of the box-like confinement and show that there exists a regime where multiple scattering of gas from the walls of the box might result in formation of "real" solitons, i.e. multi-peak structures that undergo elastic collisions. We suggest a new version of the experiment and specify physical conditions supporting the formation of "genuine" soliton trains. The underlying physics of our proposition can be understood with the help of the temporal Talbot effect, already experimentally observed in the context of Bose-Einstein condensate of sodium atoms diffracted by the pulsed grating [10].

To describe the experiment of Ref. [4] we start, as the Ref. [8] does, with the time-dependent dissipative Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{tr} + g|\psi|^2 - i\gamma|\psi|^4\right)\psi\ , \qquad (1)$$

where $\psi(\mathbf{r},t)$ is the macroscopic wave function of Bose-Einstein condensate of ^{7}Li atoms, $V_{tr}(z,\rho) = m\,(\omega_z^2 z^2 + \omega_\perp^2\rho^2)\,/2 + V_{box}$ is the axially symmetric trapping potential, $g = 4\pi\hbar^2 a_s/m$ (with a_s being the scattering length which determines the strength of the interaction). The number of atoms equals $N = 3\times 10^5$, the geometry of the harmonic confinement is given by $\omega_z = 2\pi\times 4\,\mathrm{Hz}$ and $\omega_\perp = 2\pi\times 800\,\mathrm{Hz}$, and the box-like potential V_{box} (the "end caps") of length of $L = 12$ osc. units is positioned on the slope of the harmonic potential, shifted by 15 osc. units from its center. The initial value of the scattering length of ^{7}Li condensate prepared in the end caps, which is positive and equals $200\,\mathrm{a}_0$ (a_0 – Bohr radius), is changed within approximately 10 ms to its final value $a_s = -3\mathrm{a}_0$ and then the system is kept in the box-like potential for further 17 ms. Finally, the end caps are turned off. As opposed to the Ref. [8], our calculational parameters are the same as in the experiment of Ref. [4].

The imaginary term in Eq. (1) describes the losses due to three-body recombination processes [11], [8]. Its presence is necessary to get the agreement with experimental results. Since the losses were not investigated experimentally, we follow the references just mentioned and put $\gamma = 2.05\times10^{-26}\,\mathrm{cm}^6\,\mathrm{s}^{-1}$. The dissipative term is turned on, according to the observation in [4], when the interaction strength becomes negative and is strongly diminished (by a factor of 100) when the end caps are off. When the losses are kept constant during the whole calculations the final number of atoms is much less than the number observed in the experiment. On the other hand,

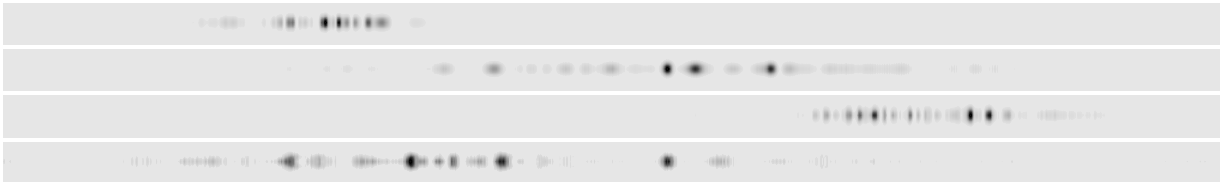

FIG. 1: Moving soliton train of ^{7}Li atoms near the turning points and at the center of the oscillation. Successive images correspond to 6 ms, 48 ms, 125 ms, and 185 ms after the end caps are off. The number of atoms, the trap parameters, and the way the scattering length is changed are the same as in the experiment of Ref. [4]. Color images are available at [12].

decreasing the dissipative term leads to the condensate collapse. Another way of getting agreement with experimental data (i.e. the final number of atoms) is to lower the initial number of atoms while keeping the same value of γ all the time. Hence, the dissipative term plays a crucial role in stabilizing the system. Certainly, it has an influence on the collapse studied recently in Ref. [9].

After switching off the box potential the bosonic cloud starts to oscillate in harmonic trap. As it is illustrated in Fig. 1, the clouds actually breaks into several peaks which propagate in the potential for many oscillatory cycles. However, we observe that certain peaks disappear during the evolution and reappear again later. For example, in the second frame of Fig. 1 when the condensate goes through the oscillatory center from the left to the right, the number of distinguishable peaks is 3, whereas in the lowest frame (the system goes now to the left) that number is increased to 4. Such structures can not be, in fact, considered as solitons.

Explaining the experimental results of Ref. [4] we, as opposed to other theoretical works [7, 8, 9], emphasize the role of the box-like potential. We demonstrate how important is the time when it is open and its location. To this end, we have performed calculations for a smaller sample of $N = 10^4$ ^{7}Li atoms created with initial scattering length $a_s = 100\mathrm{a}_0$, which is next changed within 10 ms to the final value $a_s = -3\mathrm{a}_0$. The dissipative term is kept constant all the time. Our results are summarized in Fig. 2. The end caps are positioned symmetrically with respect to the center of the harmonic trap. Only in such a configuration we discover the existence of time windows, i.e. the time intervals for the opening the end caps that support the generation of train of real solitons (multipeak structures with preserved number of peaks). This is illustrated in Fig. 2, where the size of the box-like potential is $L = 4.0$ osc. units ($75.6\,\mu$m). Frame (a) shows the density in the case when the end caps are switched off within the time window (61 ms after the interaction strength is changed). Here the number of peaks is still the same. On the contrary, when the end caps are off at the time outside the time window, the number of peaks changes resembling the missing solitons observed in the experiment [4]. Calculations also show the existence of further time windows, especially for shorter end caps. The duration of the time window is about 3 ms and increases when the size of the end caps gets larger.

It is important to understand the role of the location of the box-like potential. We have checked numerically that the time windows persist even if the end caps are shifted from the center of the harmonic potential by several percent, depending on the size of the end caps. Moreover, we do not see the time windows when the size of the end caps is too large. It is clear that the discussed phenomenon is somehow spoiled by the presence of the harmonic trap. As will be explained later, the disappearance of time windows is caused by the loss of Talbot-type recurrence due to nonuniform shift of energy levels of the box potential forced by the harmonic trap. Certainly, the most favorable conditions for the existence of time windows are those when there is no axial harmonic confinement. Such calculations have been already performed, see Fig. 1 in Ref. [8]. However, the authors of Ref. [8] have not investigated carefully the influence of the delay time between the removal of the end caps and the switching off the scattering length and they overlooked the possibility of existence of time windows. In Fig. 3 we plot the axial density profiles in the case when the box-like potential is turned off just at the time when the scattering length is changed (left column) and in the case when the box is removed within the time window. The left column shows that, in fact, the condensate is split into 6 peaks (not 5 as reported in [8]) and its evolution is violent in a sense that the number of peaks changes for a long duration. On the other hand, as demonstrated in the right column, the number of peaks (solitons) is settled much earlier when the end caps are switched off within the time window.

The origin of time windows is due to the temporal Talbot effect. Therefore, we start with a purely linear case and consider the evolution of the symmetric wave packet in an "open" rectangular one-dimensional box. The walls of the box are very high and therefore we will expand the initial wave packet in the basis of the symmetric eigenstates of infinitely deep well potential that are $\sqrt{2/L}\cos(k_n x)$ with $k_n = \frac{\pi}{L}(2n+1)$, where $n = 0, 1, ...$ and L is the length of the well. Note that the wave functions are zero in the classically forbidden region. The time evolution reads

$$\psi(x,t) = \sum_n \alpha_n e^{-iE_0(2n+1)^2 t/\hbar} \sqrt{\frac{2}{L}} \cos k_n x\,, \qquad (2)$$

where $E_0 = \hbar^2\pi^2/(2mL^2)$ and $\{\alpha_n\}$ are the coefficients of the expansion of $\psi(x,0)$. Fourier transform of (2) is

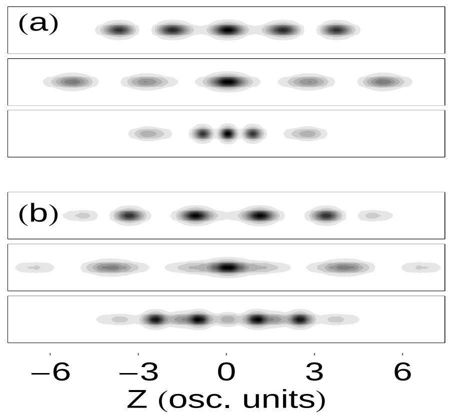

FIG. 2: Illustration of the importance of a time when the end caps are switched off. Frame (a) shows the condensate density at 30 ms, 66 ms, and 108 ms after the box is off, i.e. 61 ms after the interaction strength is changed. This is the case of the time window when, as a result, the structure consisted of the same number of peaks is formed. As opposed, frame (b) resembles Fig. 1. Here, the number of peaks changes during the evolution. Successive snapshots correspond to 30 ms, 66 ms, and 139 ms after the end caps are turned off outside the time window (52 ms after the change of the interaction strength).

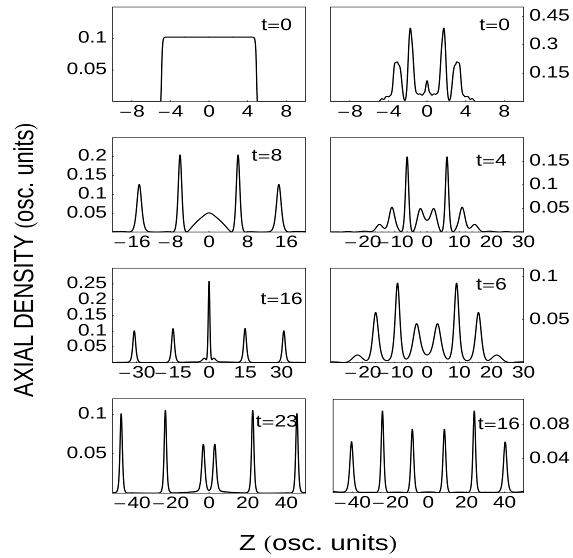

FIG. 3: Axial density profiles of a condensate of 10^4 ^{7}Li atoms during the evolution under no axial confinement. The size of the end caps equals 189 μm. The left column corresponds to Fig. 1 in Ref. [8] when time window condition is not fulfilled. For the right column the end caps are turned off within the time window (314 ms after the scattering length is changed). In both cases the time $t = 0$ means the time when the end caps are removed.

given by

$$\tilde{\psi}(k,t) = \sqrt{\frac{L}{2}} \sum_n \alpha_n e^{-iE_0(2n+1)^2 t/\hbar} \times \left(\frac{\sin \frac{L}{2}(k-k_n)}{\frac{L}{2}(k-k_n)} + \frac{\sin \frac{L}{2}(k+k_n)}{\frac{L}{2}(k+k_n)} \right) . \quad (3)$$

According to the above formula the momentum distribution is recovered after a period of $T_{rev} = \pi/4$ (in units of $\hbar/E_0$) and this is the result of the Talbot-type recurrence occurring in the phase factor ("$n(n+1)$" dependence on the quantum number n). However, this formula exhibits more structure. At time $T_{win}^{lin} = \pi/8$ the resulting momentum distribution forms a set of fully separated groups centered at momenta $k = k_n$. This can be verified (e.g. numerically) taking into account several facts: (1) the form of the phase factor in Eq. (3); (2) the localization of the function $\sin(x)/x$ around $x = 0$; (3) assumed weak dependence of coefficients α_n on n.

If the box-like potential is switched off at times around $T_{win}^{lin} = \pi/8$ all groups of momenta transform into separated wave packets which just spread out during the further evolution. This spreading can be stopped by turning on the nonlinear term and then the wave packets are transformed into Zakharov solitons [1]. Hence, the nonlinearity is essential to built the solitons from the separated wave packets. Here, the separated wave packets originate from the temporal Talbot effect combined with the appropriate initial conditions. In a real experiment initial conditions are formed as a result of modulational instability driven by the condensate collapse.

Looking at the formula (3), it is clear that the presence of the box-like potential is essential since it allows the condensate wave function for multiple reflexions from the walls resulting in the interference pattern found above. However, "small addition" of the harmonic potential does not spoil the picture. This can be proved by the perturbation calculus. It turns out that for not too large size of the box L ($L < 5$ osc. units) all the levels but the lowest one are shifted approximately uniformly. In such a way the new time scale that is proportional to the reciprocal of the difference of the shifts of the two lowest levels (and approximately equals the trap period) can be introduced. Based on the formula (3) only those time windows survive that occur at times shorter than the trap period (although "revivals" are observed for long enough times).

We can also prove that the temporal Talbot effect survives under the presence of the weak nonlinearity. For small enough attraction the nonlinear term that appears in the Gross-Pitaevskii equation can be treated as a perturbation to the linear case. It turns out that all single-particle levels are shifted up uniformly. Therefore, from the formula (3), one should expect formation of groups in momentum space at the same times and of the same duration as in the linear case.

The temporal Talbot effect persists also when the nonlinearity is larger and comparable to that one present in the experiment of Ref. [4]. As it is the case, we plot in Fig. 4 the axial momentum densities (at the time when the end caps are off) corresponding to the frames

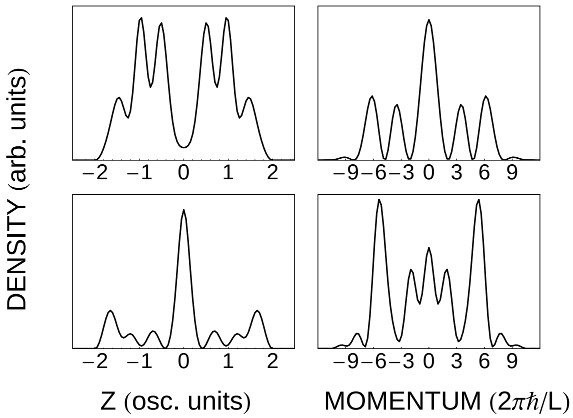

FIG. 4: Spatial (left column) and momentum (right column) densities at the time when the end caps are off. The upper frames illustrate the case of the time window in which the momentum density shows well developed groups of momenta. These groups are next transformed to groups in a position space. The lower frames correspond to the case when there are no separated peaks in a momentum space and consequently the probability flow leads to "missing solitons" structures.

(a) and (b) of Fig. 2. Fig. 4 proves that the time windows survive under the conditions when both the box-like and harmonic confinements and the nonlinearity are present. When the end caps are off just within the time window (the upper frames) the momentum density shows five distinguishable peaks whereas the density in the position space consists of two broad peaks with partially developed three subpeaks. Upper frames in Fig. 2 confirm that afterward five (not two nor six) solitons are developed. It means that all momenta groups have been transformed to peaks in position space. These peaks oscillate with the trap period, collide when they meet at the center of the trap and then reappear. Their motion is particle-like. If one considers a point-like particle moving according to the Newton equation, initially placed at the trap center and assign the initial velocity determined by the maximum value of the momenta group, the particle will follow the density peak. Hence, it is reasonable to use the name solitons for the density peaks.

When the end caps are switched off outside the time window (lower frames in Fig. 4) the number of density peaks changes during the evolution (see the lower part of Fig. 2). This is because the momentum density does not consist of well separated groups and the probability can flow from one group to the other forming the "missing solitons" structures. This is, in fact, the case of numerical calculations reported in Ref. [8]. Fig. 5 of Ref. [8] shows that the number of peaks changes in time while the condensate is axially confined. On the other hand, when the axial confinement is off the condensate always ends in a state with a fixed number of solitons.

The time the first time window appears is slightly larger than $T_{win}^{lin} = \pi/8$ predicted from the linear theory. We found that it is shifted by an amount of the order of characteristic time scale $\hbar/(gN/V)$, which in our case is equal to several milliseconds.

In conclusion, we have shown the importance of the box-like potential in experiments like that of Ref. [4]. Contrary to missing solitons structures genuine bright solitons can be generated in a repulsive condensate by changing the sign of the scattering length with the help of the Feshbach resonance technique and keeping the condensate in the end caps for long enough time. In fact, what we propose to create solitons is to utilize the Talbot effect that is well known from the linear physics. It turns out, however, that this effect survives under the condition of weak nonlinearity what was proved by using the perturbative analysis but also under the conditions of the experiment of Ref. [4] as was demonstrated by direct solving of the Gross-Pitaevskii equation. Perhaps, it is not surprising since the temporal Talbot effect was already experimentally observed for a condensate of sodium atoms, although in a different situation when the condensate was diffracted by a pair of pulsed gratings and the periodicity with respect to the time delay between pulses in a series of condensate images was discovered [10].

Acknowledgments

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